

# Ribet's modular construction of unramified $p$ -extensions of $\mathbf{Q}_{(p)\mu}$

*Notes for a series of lectures delivered at Gauhati*

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*Ribet's ingenious method is the gateway to much later progress.*

— Barry Mazur

For every prime  $p$ , we have the cyclotomic number field  $K = \mathbf{Q}(\zeta)$ , obtained by adjoining to  $\mathbf{Q}$  a primitive  $p$ -th root  $\zeta$  of 1; it is a Galois extension of  $\mathbf{Q}$ , cyclic of degree  $p-1$ . The ideal class group  $A$  of  $K$  is a finite commutative group which measures the failure of the ring of integers of  $K$  to be factorial or principal. We are interested in the  $\mathbf{F}_p$ -space  $C = A/A^p$ . As  $A$  and  $C$  are trivial for  $p = 2$ , we assume that the prime  $p$  is *odd*.

In the middle of the XIX<sup>th</sup> century, Kummer made a deep connection between the group  $C$  and the sequence of Bernoulli numbers  $(B_k)_{k \in \mathbf{N}}$  defined by

$$\frac{T}{e^T - 1} = \sum_{k \in \mathbf{N}} B_k \frac{T^k}{k!},$$

so that  $B_0 = 1$ ,  $B_1 = -1/2$ . As the function  $\frac{T}{e^T - 1} - 1 + \frac{T}{2}$  is even (invariant under  $T \mapsto -T$ ), we have  $B_k = 0$  for  $k$  odd  $> 1$ . Here are the numerators  $N_k$  and the denominators  $D_k$  of the first few Bernoulli numbers  $B_k = N_k/D_k$  of even index  $k > 0$ :

$k =$	2	4	6	8	10	12	14	16	18	20
$N_k =$	1	-1	1	-1	5	-691	7	-3617	43867	-174611
$D_k =$	6	30	42	30	66	2730	6	510	798	330

Kummer's criterion says that *the  $\mathbf{F}_p$ -space  $C$  is  $\neq 0$  if and only if  $p$  divides (the numerator of) one of the rational numbers  $B_2, B_4, \dots, B_{p-3}$* . This is equivalent (see §6) to  $p$  dividing the product  $B_2 B_4 \dots B_{p-3}$ ; the sequence of such *irregular* primes is 37 (which divides  $B_{32} = 7709321041217/510$ ), 59| $B_{44}$ , 67| $B_{58}$ , 101| $B_{68}$ , 103| $B_{24}$ , ..., 691| $B_{12}$ , ...

The  $\mathbf{F}_p$ -space  $C$  carries an action of the group  $\Delta = \text{Gal}(K|\mathbf{Q})$ , for which there is a canonical isomorphism  $\chi : \Delta \rightarrow \mathbf{F}_p^\times$ ; its inverse sends  $a \in \mathbf{F}_p^\times$  to the automorphism  $\sigma_a : \zeta \mapsto \zeta^a$  of  $K$ . Since every character  $\Delta \rightarrow \bar{\mathbf{F}}_p^\times$  is of the form  $\chi^i$  for a unique  $i \in \mathbf{Z}/(p-1)\mathbf{Z}$ , we have a direct sum decomposition  $C = \oplus_{i \in \mathbf{Z}/(p-1)\mathbf{Z}} C(\chi^i)$ , where

$$C(\chi^i) = \{c \in C \mid \sigma(c) = \chi^i(\sigma)c \text{ for every } \sigma \in \Delta\}.$$

In the early 30s, Herbrand refined one of the implications in Kummer's criterion. He showed that, *if  $k \in [2, p-3]$  is an even integer such that  $C(\chi^{1-k}) \neq 0$ , then  $p$  divides (the numerator of)  $B_k$* . This is an easy consequence of Stickelberger's theorem [18, p. 101].

The main result in Ribet's paper is the converse of Herbrand's theorem. In other words, he showed that *if  $p$  divides  $B_k$  for some even integer  $k \in [2, p-3]$ , then  $C(\chi^{1-k}) \neq 0$* .

(We may allow  $k = p-1$  in the statement “ $p|B_k \Leftrightarrow C(\chi^{1-k}) \neq 0$ ” of the Herbrand-Ribet theorem. We shall see in §6 that in fact  $p$  always divides the *denominator* of  $B_{p-1}$ . We must therefore show that  $C(\chi) = 0$ . This follows from Kummer theory and the fact that the only degree- $p$  cyclic extension of  $\mathbf{Q}$  unramified outside  $p$  is the one contained in  $\mathbf{Q}(\sqrt[p]{\zeta})$ .)

Let us end this introduction by mentioning how the Main Conjecture (for the cyclotomic  $\mathbf{Z}_p$ -extension of  $\mathbf{Q}$ ), as first proved by Mazur-Wiles, implies a quantitative generalisation of the Herbrand-Ribet theorem; see Ribet's own account [17], also available on his homepage.

Instead of the group  $C = A \otimes \mathbf{F}_p$ , consider the  $p$ -part  $D = A \otimes \mathbf{Z}_p$  of the ideal class group  $A$  of  $K = \mathbf{Q}(\zeta)$ . As for  $C$ , we have the decomposition  $D = \bigoplus_{i \in \mathbf{Z}/(p-1)\mathbf{Z}} D(\omega^i)$  as a  $\Delta$ -module, where  $\omega : \Delta \rightarrow \mathbf{Z}_p^\times$  is obtained from  $\chi$  by composing with the canonical (“Teichmüller”) section  $\mathbf{F}_p^\times \rightarrow \mathbf{Z}_p^\times$  of the projection  $\mathbf{Z}_p^\times \rightarrow \mathbf{F}_p^\times$ ; every character  $\Delta \rightarrow \bar{\mathbf{Q}}_p^\times$  is of the form  $\omega^i$  for a unique  $i \in \mathbf{Z}/(p-1)\mathbf{Z}$ . The Herbrand-Ribet theorem can be restated as “ $p|B_k \Leftrightarrow D(\omega^{1-k}) \neq 0$ ”.

The Main Conjecture implies that,  $k \in [2, p-3]$  being an even integer, *the group  $D(\omega^{1-k})$  has the same order as  $\mathbf{Z}_p/L(\omega^{k-1}, 0)\mathbf{Z}_p$* , where, for any power  $\varepsilon$  of  $\omega$ ,

$$L(\varepsilon, 0) = \frac{-1}{p} \sum_{a=1}^{p-1} \varepsilon(\sigma_a) a, \quad (\sigma_a : \zeta \mapsto \zeta^a).$$

Because of the Kummer congruence  $L(\omega^{k-1}, 0) \equiv -B_k/k \pmod{p}$  [18, p. 61], we have  $p|B_k \Leftrightarrow p|L(\omega^{k-1}, 0)$ , so Herbrand-Ribet is indeed a consequence of Mazur-Wiles. (There is as yet no unconditional proof that the groups  $D(\omega^{1-k})$  are cyclic.)

Although it is now possible, after the work of Thaine and Kolyvagin, to give a more elementary proof [18, Chap. 15] of Herbrand-Ribet, and indeed of Mazur-Wiles, the original approach using Galois representations attached to modular forms was a major source of inspiration for much subsequent work and remains of paramount importance today.

Our aim here is to provide the reader with an overview of Ribet's

modular construction (§1–4); the numbering of the statements is the same as in [16]. §0 contains some results of a general nature which are needed in the proof. §5 mentions an alternative approach which can now be carried out. Finally, §6 gives Witt’s proof of the von Staudt–Clausen theorem on the denominators of  $B_k$ .

## 0. Prerequisites

In this section we recall, without proof but with references to the literature, a certain number of results which Ribet needs in order to carry out his construction. The reader may go directly to §1–4 and consult this § only if he needs to look up the statement of a result being used in the proof.

*Equivariance of the reciprocity map.* — First, a special case of a certain functorial property of the Artin reciprocity map. Let  $M|K|\mathbf{Q}$  be finite extensions such that  $M|\mathbf{Q}$  is Galois with group  $G$ , such that  $K|\mathbf{Q}$  is Galois with group  $\Delta$ , and such that the Galois group  $H \subset G$  of  $M|K$  is commutative. Denote by  $\mathcal{C}_K$  the idèle class group of  $K$ , and by  $\psi : \mathcal{C}_K \rightarrow H$  the reciprocity map. The group  $\Delta$  acts on  $\mathcal{C}_K$  as well as on  $H$ , by conjugation on the latter. We will need the fact that  $\psi$  is equivariant for these actions :

**THEOREM 0.1** [3, p. 199]. — *For every  $\sigma \in G$  (with image  $\bar{\sigma} \in \Delta$ ) and every  $x \in \mathcal{C}_K$ , we have  $\psi(\bar{\sigma}x) = \sigma\psi(x)\sigma^{-1}$  in  $H$ .*

*Locally algebraic characters.* — Let  $\bar{\mathbf{Q}}$  be an algebraic closure of  $\mathbf{Q}$  and put  $G_{\bar{\mathbf{Q}}} = \text{Gal}(\bar{\mathbf{Q}}|\mathbf{Q})$ . We refer the reader to [14, p. III-2] for the definition of a *locally algebraic* abelian representation  $G_{\bar{\mathbf{Q}}} \rightarrow \text{GL}_n(K)$  where  $K$  is now a finite extension of  $\mathbf{Q}_p$ . Denote the  $p$ -adic cyclotomic character by  $\tilde{\chi} : G_{\bar{\mathbf{Q}}} \rightarrow \mathbf{Z}_p^\times \rightarrow K^\times$ .

**PROPOSITION 0.2.** — *Every locally algebraic character  $\varphi : G_{\bar{\mathbf{Q}}} \rightarrow K^\times$  is of the form  $\varphi = \tilde{\chi}^n \omega$  for some  $n \in \mathbf{Z}$  and some character  $\omega : G_{\bar{\mathbf{Q}}} \rightarrow K^\times$  of finite order.*

*The semisimplification of the reduction of a  $p$ -adic representation.* — The next result says that the reduction of a  $p$ -adic representation may depend upon the choice of a stable lattice, but the semisimplification does not. More precisely, let  $K$  be a finite extension of  $\mathbf{Q}_p$ ,  $\mathfrak{o}$  the ring of integers of  $K$ ,  $\mathfrak{p}$  the unique maximal ideal of  $\mathfrak{o}$ , and  $F = \mathfrak{o}/\mathfrak{p}$  the residue field. Recall that an  $\mathfrak{o}$ -lattice in a finite-dimensional  $K$ -space  $V$  is a sub- $\mathfrak{o}$ -module generated by any  $K$ -basis of  $V$ .

Let  $G$  be a profinite group — a compact totally disconnected group — and  $\rho : G \rightarrow \text{GL}(V)$  a (continuous) representation. Among the  $\mathfrak{o}$ -lattices

in  $V$ , there is at least one which is  $G$ -stable, for if  $\Lambda \subset V$  is any  $\mathfrak{o}$ -lattice, the subgroup  $H = \rho^{-1}(\mathrm{GL}(\Lambda))$  is open in  $G$ , hence the index  $(G : H)$  is finite, and the  $\mathfrak{o}$ -lattice  $\sum_{\sigma \in G/H} \sigma(\Lambda)$  is  $G$ -stable.

The choice of a  $G$ -stable  $\mathfrak{o}$ -lattice  $\Lambda \subset V$  leads to a (continuous) representation  $\rho_\Lambda : G \rightarrow \mathrm{GL}(\Lambda) \rightarrow \mathrm{GL}(\Lambda/\mathfrak{p}\Lambda)$  on a finite-dimensional  $F$ -space, called the reduction of  $\rho$  along  $\Lambda$ . The theorem of Brauer-Nesbitt says that

PROPOSITION 0.3 [4, p. 215]. — *The semisimplification  $\bar{\rho}$  of  $\rho_\Lambda$  depends only on  $\rho$ , not on the choice of  $\Lambda$ .*

*The representation associated to a weight-2 level- $p$  eigenform.* — Next, we need some facts about modular forms. Let  $f \in \mathfrak{S}_2(\Gamma_1(p))$  be a normalised cuspidal eigenform of weight 2 for  $\Gamma_1(p)$ , and write the  $q$ -expansion (with  $q = e^{2i\pi\tau}$  and  $\tau \in \mathfrak{H}$ ) of  $f$  as  $f = \sum_{n \geq 0} a_n q^n$ .

PROPOSITION 0.4 [8, p. 234]. — *The subfield  $K_f = \mathbf{Q}(a_1, a_2, \dots)$  of  $\mathbf{C}$  is of finite degree over  $\mathbf{Q}$ . Moreover, if  $\varepsilon : (\mathbf{Z}/p\mathbf{Z})^\times \rightarrow \mathbf{C}^\times$  is the character such that  $f \in \mathfrak{S}_2(p, \varepsilon)$ , then the field  $\mathbf{Q}(\varepsilon)$  generated by the values of  $\varepsilon$  is contained in the number field  $K_f$ .*

Recall the construction of the (2-dimensional) representations  $\rho_{f, \mathfrak{p}} : G_{\bar{\mathbf{Q}}} \rightarrow \mathrm{GL}(V_{f, \mathfrak{p}})$  attached to a normalised eigenform  $f \in \mathfrak{S}_2(p, \varepsilon)$ . Here  $\mathfrak{p}$  is any prime of  $K_f$  and  $V_{f, \mathfrak{p}}$  is a certain vector space — described below — over the completion  $K_{f, \mathfrak{p}}$  of  $K_f$  at the prime  $\mathfrak{p}$ . Shimura attaches to  $f$  an abelian variety  $A_f$  over  $\mathbf{Q}$  (uniquely determined up to  $\mathbf{Q}$ -isogeny) with the following properties.

The abelian variety  $A_f$  has dimension  $\dim A_f = [K_f : \mathbf{Q}]$ , it is a quotient of the jacobian  $J_1(p) = \mathrm{Jac} X_1(p)$  of the modular curve  $X_1(p)$ , and there is an embedding of  $\mathbf{Q}$ -algebras  $K_f \rightarrow \mathrm{End}_{\mathbf{Q}}(A_f) \otimes \mathbf{Q}$ .

Let  $T_p(A_f) = \lim_{n \in \mathbf{N}^*} {}_p A_f(\bar{\mathbf{Q}})$  be the  $p$ -adic Tate module of  $A_f$  and  $V_p(A_f) = T_p(A_f) \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$ , which is thus a module over  $K_f \otimes \mathbf{Q}_p$ .

PROPOSITION 0.5 [8, p. 390]. — *The  $(K_f \otimes \mathbf{Q}_p)$ -module  $V_p(A_f)$  is free of rank 2.*

It follows that  $V_{f, \mathfrak{p}} = V_p(A_f) \otimes_{K_f \otimes \mathbf{Q}_p} K_{f, \mathfrak{p}}$  is a 2-dimensional vector space over  $K_{f, \mathfrak{p}}$  with a continuous linear action of  $G_{\bar{\mathbf{Q}}}$ . Denote this degree-2 representation by

$$\rho_{f, \mathfrak{p}} : G_{\bar{\mathbf{Q}}} \rightarrow \mathrm{GL}(V_{f, \mathfrak{p}}).$$

The abelian variety  $A_f$  has good reduction at every prime  $l \neq p$  of  $\mathbf{Q}$ . Further, the Eichler-Shimura relations give

PROPOSITION 0.6 [8, p. 390]. — *The representation  $\rho_{f,p}$  is unramified at every prime  $l \neq p$  and the trace (resp. determinant) of a Frobenius element  $\text{Frob}_l$  at  $l$  is  $a_l$  (resp.  $l\varepsilon(l)$ ) in  $K_{f,p}$  (resp.  $K_{f,p}^\times$ ).*

The final bit of information about  $A_f$  that we need is the following theorem of Deligne-Rapoport.

THEOREM 0.7 [6, p. 113]. — *Suppose that  $\varepsilon$  is not trivial. The abelian variety  $A_f$  acquires good reduction at the unique prime dividing  $p$  in the maximal real subfield  $\mathbf{Q}(\zeta)^+$  of the cyclotomic field  $\mathbf{Q}(\zeta)$  (where  $\zeta^p = 1$ ,  $\zeta \neq 1$ ).*

*Finite flat group schemes.* — Let  $E|\mathbf{Q}_p$  be a finite extension, with ring of integers  $\mathfrak{o}_E$ . Ribet makes crucial use of the following result of Raynaud.

THEOREM 0.8 [15, p. 268]. — *Suppose that the ramification index of  $L|\mathbf{Q}_p$  is  $< p - 1$ . Let  $G$  be a finite flat commutative group scheme over  $L$ , killed by a power of  $p$ . There is at most one finite flat extension of  $G$  to  $\mathfrak{o}_L$ .*

## 1. The main theorem

Adopt the same notation as in the introduction. In particular,  $p$  is an odd prime and  $K = \mathbf{Q}(\zeta)$ , with  $\zeta^p = 1$ ,  $\zeta \neq 1$ .

THEOREM 1.1. — *Let  $k \in [2, p - 3]$  be an even integer. If  $p|B_k$ , then  $C(\chi^{1-k}) \neq 0$ .*

By the functorial property (0.1) of the Artin symbol, it is sufficient to construct a suitable everywhere-unramified extension  $E|K$ .

THEOREM 1.2. — *Let  $k \in [2, p - 3]$  be an even integer, and suppose that  $p|B_k$ . There exists a Galois extension  $E|\mathbf{Q}$  containing  $K$  such that*

- (a) *The extension  $E|K$  is everywhere unramified.*
- (b) *The group  $H = \text{Gal}(E|K)$  is a  $p$ -elementary commutative group ( $\neq 1$ ); equivalently,  $H$  is an  $\mathbf{F}_p$ -space of dimension  $> 0$ .*
- (c) *For every  $\sigma$  in  $G = \text{Gal}(E|\mathbf{Q})$  (with image  $\bar{\sigma}$  in  $\Delta$ ) and every  $\tau$  in  $H$ ,*

$$\sigma\tau\sigma^{-1} = \chi(\bar{\sigma})^{1-k}.\tau$$

*Proof that (1.2)  $\Rightarrow$  (1.1).* Let  $M$  be the maximal  $p$ -elementary abelian extension of  $K$  which is unramified at every place of  $K$ ; by maximality,  $M|\mathbf{Q}$  is Galois. The Artin map  $\psi : \mathcal{C}_K \rightarrow \text{Gal}(M|K)$ , where  $\mathcal{C}_K$  is the idèle class group of  $K$ , is a homomorphism inducing an isomorphism

$C \rightarrow \text{Gal}(M|K)$ . Moreover, for every  $\sigma \in \text{Gal}(M|\mathbf{Q})$  (with image  $\bar{\sigma} \in \Delta$ ) and every  $x \in \mathcal{C}_K$ , we have  $\psi(\bar{\sigma}x) = \sigma\psi(x)\sigma^{-1}$  (0.1).

In other words, the isomorphism (of  $\mathbf{F}_p$ -spaces)  $C \rightarrow \text{Gal}(M|K)$  induced by  $\psi$  is  $\Delta$ -equivariant, and hence induces, for every  $i \in \mathbf{Z}/(p-1)\mathbf{Z}$ , an isomorphism  $C(\chi^i) \rightarrow \text{Gal}(M|K)(\chi^i)$ . If (1.2) holds, the latter group is  $\neq 0$  for  $i = 1 - k$ , because it has the quotient  $H$ . It follows that if  $p|B_k$ , then  $C(\chi^{1-k}) \neq 0$ , so (1.1) holds.

The next theorem shows how such an  $E$  can be constructed from a certain kind of representation of  $G_{\bar{\mathbf{Q}}}$ , where  $\bar{\mathbf{Q}}$  is an algebraic closure of  $\mathbf{Q}$  containing  $K$ . Choose a decomposition group  $D_p \subset G_{\bar{\mathbf{Q}}}$  at the prime  $p$ , and let  $\chi$  also stand for the composite  $G_{\bar{\mathbf{Q}}} \rightarrow \Delta \rightarrow \mathbf{F}_p^\times$ .

**THEOREM 1.3.** — *Let  $k \in [2, p-3]$  be even, and suppose that  $p|B_k$ . There exists a finite extension  $F|\mathbf{F}_p$ , and a (continuous) representation  $r : G_{\bar{\mathbf{Q}}} \rightarrow \text{GL}_2(F)$ , such that*

- (i)  *$r$  is unramified at every prime  $l \neq p$ .*
- (ii)  *$r$  is an extension of  $\chi^{k-1}$  by the trivial 1-dimensional representation :*

$$r \sim \begin{pmatrix} 1 & \gamma \\ 0 & \chi^{k-1} \end{pmatrix}$$

*for some map  $\gamma : G_{\bar{\mathbf{Q}}} \rightarrow F$ .*

- (iii) *The order of  $\text{Im}(r)$  is divisible by  $p$ .*
- (iv) *The order of  $\text{Im}(r|_{D_p})$  is not divisible by  $p$ .*

*Proof that (1.3)  $\Rightarrow$  (1.2).* Let  $r$  be a representation as in (1.3); we have  $\det r = \chi^{k-1}$ , and there is a tower of fields  $E'|K'|\mathbf{Q}$ , with  $E'$  the field fixed by  $\text{Ker } r$  and  $K'$  the field fixed by  $\text{Ker } \chi^{k-1}$ ; we have  $K' \subset K$ , because  $\text{Ker } \chi \subset \text{Ker } \chi^{k-1}$ . Put  $H' = \text{Gal}(E'|K')$ , so that the sequence  $1 \rightarrow H' \rightarrow \text{Im } r \rightarrow \text{Im } \chi^{k-1} \rightarrow 1$  is exact. Notice that  $H'$  is a  $p$ -elementary commutative group, for  $\gamma$  gives rise to an injective homomorphism  $H' \rightarrow F$ , and its order is  $> 1$ , for the order of  $\text{Im } r$  is divisible by  $p$ , by (1.3)(iii).

Next, remark that  $E'|K'$  is everywhere unramified. Indeed, for every prime  $l \neq p$ , the extension  $E'|\mathbf{Q}$  is unramified at  $l$  by (1.3)(i), and hence so is  $E'|K'$  at every prime  $l$  of  $K'$  prime to  $p$ . As for the unique prime  $\mathfrak{p}$  of  $K'$  dividing  $p$ , the inertia subgroup at  $\mathfrak{p}$  has order prime to  $p$  by (1.3)(iv), so  $E'|K'$  is at worst tamely ramified, and hence unramified, because  $E'|K'$  is a  $p$ -extension.

Taking  $E = E'K$ , it is now clear that (1.2)(a) and (1.2)(b) hold for  $E|K|\mathbf{Q}$ , because the analogous statements hold for  $E'|K'|\mathbf{Q}$ , as we have

just seen. It remains to show that (1.2)(c) holds as well. This follows from the fact that for every  $\sigma, \tau \in G_{\mathbf{Q}}$ , we have

$$\begin{pmatrix} 1 & \gamma(\sigma) \\ 0 & \chi^{k-1}(\sigma) \end{pmatrix} \begin{pmatrix} 1 & \gamma(\tau) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \gamma(\sigma) \\ 0 & \chi^{k-1}(\sigma) \end{pmatrix}^{-1} = \begin{pmatrix} 1 & \chi^{1-k}(\sigma) \cdot \gamma(\tau) \\ 0 & 1 \end{pmatrix},$$

because of the identity  $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}^{-1} = \begin{pmatrix} 1 & ad^{-1} \cdot x \\ 0 & 1 \end{pmatrix}$ .

It remains to find such an  $r$ . We shall take  $r$  to be a suitable reduction (§2) of the representation  $\rho_{f,\mathfrak{p}}$  (§4) attached to a certain weight-2 level- $p$  cuspidal eigenform  $f$  (§3).

## 2. Reductions of $p$ -adic representations

Let  $K|\mathbf{Q}_p$  be a finite extension,  $\mathfrak{o}$  its ring of integers,  $\mathfrak{p}$  the maximal ideal of  $\mathfrak{o}$ , and  $F$  the residue field. Let  $V$  be a 2-dimensional  $K$ -space with a continuous linear action  $\rho$  of a profinite group  $G$ , and suppose that  $\bar{\rho}$  (0.3) is reducible. The two characters  $\varphi_1, \varphi_2 : G \rightarrow F^\times$  such that  $\bar{\rho} \sim \varphi_1 \oplus \varphi_2$  are uniquely determined by  $\rho$  up to order. Also, for every  $G$ -stable  $\mathfrak{o}$ -lattice  $\Lambda \subset V$ , the reduction  $\rho_\Lambda$  is reducible, and its semisimplification is isomorphic to  $\varphi_1 \oplus \varphi_2$  (0.3). For a given  $\Lambda$ ,  $\rho_\Lambda$  is isomorphic to one of

$$\begin{pmatrix} \varphi_1 & * \\ 0 & \varphi_2 \end{pmatrix}, \quad \begin{pmatrix} \varphi_1 & 0 \\ * & \varphi_2 \end{pmatrix},$$

and  $\rho_\Lambda$  is semisimple if  $\rho_\Lambda(G)$  has order prime to  $p$ . Ribet's lemma says that each of these two types can be realised for some  $\Lambda$ , and, if  $\rho$  is simple (but  $\bar{\rho}$  is not), then there is a  $\rho_\Lambda$  which is not semisimple and has any of the above types prescribed in advance.

**PROPOSITION 2.1.** — *Suppose that the degree-2 representation  $V$  of  $G$  is simple but  $\bar{\rho}$  is not simple. Then, for any ordering  $\varphi_1, \varphi_2$  of the two characters of which  $\bar{\rho}$  is the direct sum, there is a  $G$ -stable  $\mathfrak{o}$ -lattice  $\Lambda \subset V$  such that  $\rho_\Lambda \sim \begin{pmatrix} \varphi_1 & * \\ 0 & \varphi_2 \end{pmatrix}$ , as opposed to  $\begin{pmatrix} \varphi_1 & 0 \\ * & \varphi_2 \end{pmatrix}$ , and such that  $\rho_\Lambda$  is not semisimple — not isomorphic to  $\begin{pmatrix} \varphi_1 & 0 \\ 0 & \varphi_2 \end{pmatrix}$ .*

Choosing a  $G$ -stable  $\mathfrak{o}$ -lattice  $\Lambda \subset V$  and fixing an  $\mathfrak{o}$ -basis of  $\Lambda$  allows one to view  $\rho$  as a representation  $G \rightarrow \mathrm{GL}_2(\mathfrak{o})$ ; the reduction of  $\rho$  along  $\Lambda$  is then the composite  $\rho_\Lambda : G \rightarrow \mathrm{GL}_2(\mathfrak{o}) \rightarrow \mathrm{GL}_2(F)$ ; we call it the reduction of  $\rho(G)$ . For every  $M \in \mathrm{GL}_2(K)$  such that  $M\rho(G)M^{-1} \subset \mathrm{GL}_2(\mathfrak{o})$ , we get another  $G$ -stable  $\mathfrak{o}$ -lattice in  $V$  together with an  $\mathfrak{o}$ -basis, giving another

reduction  $G \rightarrow M\rho(G)M^{-1} \rightarrow \mathrm{GL}_2(\mathfrak{o}) \rightarrow \mathrm{GL}_2(F)$  of  $\rho$ . We will show that at least one of these reductions is of the desired type  $\begin{pmatrix} \varphi_1 & * \\ 0 & \varphi_2 \end{pmatrix}$ , and that, if all reductions of this type are semisimple, then  $\rho$  cannot be simple, which it is by hypothesis.

Let  $\pi$  be an  $\mathfrak{o}$ -basis of  $\mathfrak{p}$ , also called a uniformiser of  $K$ , and put  $P = \begin{pmatrix} 1 & 0 \\ 0 & \pi \end{pmatrix}$ ; we have

$$P \begin{pmatrix} a & \pi b \\ c & d \end{pmatrix} P^{-1} = \begin{pmatrix} a & b \\ \pi c & d \end{pmatrix}.$$

This implies that there is at least one  $G$ -stable  $\mathfrak{o}$ -lattice in  $V$  along which the reduction of  $\rho$  is of the desired type  $\begin{pmatrix} \varphi_1 & * \\ 0 & \varphi_2 \end{pmatrix}$ , for if the reduction of  $\rho(G)$  is of the other type  $\begin{pmatrix} \varphi_1 & 0 \\ * & \varphi_2 \end{pmatrix}$ , then the reduction of  $P\rho(G)P^{-1}$  is of the desired type, as the above formula shows.

Suppose therefore that the reduction of  $\rho(G)$  is of the desired type  $\begin{pmatrix} \varphi_1 & * \\ 0 & \varphi_2 \end{pmatrix}$ ; it remains to show that, among the reductions of  $\rho$  of this type, there is at least one which is not semisimple. Suppose, if possible, that every reduction of the type  $\begin{pmatrix} \varphi_1 & * \\ 0 & \varphi_2 \end{pmatrix}$  is semisimple. We shall construct a sequence of matrices  $(M_i)_{i \in \mathbf{N}}$  in  $\mathrm{GL}_2(\mathfrak{o})$ , having a limit  $M \in \mathrm{GL}_2(\mathfrak{o})$ , such that  $M\rho(G)M^{-1} \subset \begin{pmatrix} \mathfrak{o}^\times & 0 \\ \mathfrak{p} & \mathfrak{o}^\times \end{pmatrix}$ , showing that  $\rho$  is not simple — a contradiction.

Suppose that, for some  $i \in \mathbf{N}$ , a matrix  $M_i = \begin{pmatrix} 1 & t_i \\ 0 & 1 \end{pmatrix}$  in  $\mathrm{GL}_2(\mathfrak{o})$  has been found such that

$$M_i \rho(G) M_i^{-1} \subset \begin{pmatrix} \mathfrak{o}^\times & \mathfrak{p}^i \\ \mathfrak{p} & \mathfrak{o}^\times \end{pmatrix} ;$$

for  $i = 0$ , take  $M_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ . The above inclusion can be rewritten  $P^i M_i \rho(G) M_i^{-1} P^{-i} \subset \begin{pmatrix} \mathfrak{o}^\times & \mathfrak{o} \\ \mathfrak{p}^{i+1} & \mathfrak{o}^\times \end{pmatrix}$ , whose reduction is of the desired type  $\begin{pmatrix} \varphi_1 & * \\ 0 & \varphi_2 \end{pmatrix}$ , because the reduction of  $\rho(G)$  is of this type. As all reductions of this type are semisimple by hypothesis, there exists a matrix



$U_i = \begin{pmatrix} 1 & u_i \\ 0 & 1 \end{pmatrix}$  in  $\mathrm{GL}_2(\mathfrak{o})$  which diagonalises the reduction in the sense that

$$U_i P^i M_i \rho(G) M_i^{-1} P^{-i} U_i^{-1} \subset \begin{pmatrix} \mathfrak{o}^\times & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{o}^\times \end{pmatrix}.$$

Denoting the entries of a matrix  $() \in \mathrm{GL}_2(\mathfrak{o})$  by  $(a_{11}(), a_{12}(); a_{21}(), a_{22}())$ , one has  $a_{21}(U_i M U_i^{-1}) = a_{21}(M)$ . This means that we actually have

$$U_i P^i M_i \rho(G) M_i^{-1} P^{-i} U_i^{-1} \subset \begin{pmatrix} \mathfrak{o}^\times & \mathfrak{p} \\ \mathfrak{p}^{i+1} & \mathfrak{o}^\times \end{pmatrix},$$

for the  $a_{21}$  of the matrix being conjugated by  $U_i$  was in  $\mathfrak{p}^{i+1}$ . Conjugating by  $P^{-i}$  and taking  $M_{i+1} = P^{-i} U_i P^i M_i = \begin{pmatrix} 1 & t_i + u_i \pi^i \\ 0 & 1 \end{pmatrix}$ , the above inclusion implies

$$M_{i+1} \rho(G) M_{i+1}^{-1} \subset \begin{pmatrix} \mathfrak{o}^\times & \mathfrak{p}^{i+1} \\ \mathfrak{p} & \mathfrak{o}^\times \end{pmatrix},$$

completing the construction of the sequence  $(M_i)_{i \in \mathbf{N}}$  by induction. It is clear that this sequence converges in  $\mathrm{GL}_2(\mathfrak{o})$ , and that the limit  $M$  satisfies  $M \rho(G) M^{-1} \subset \begin{pmatrix} \mathfrak{o}^\times & 0 \\ \mathfrak{p} & \mathfrak{o}^\times \end{pmatrix}$ , which contradicts the simplicity of  $\rho$  and thereby proves that  $\rho$  has at least one reduction which has the desired type but which is not semisimple.

For a generalisation of (2.1), see [1].

### 3. A weight-2 level- $p$ cuspidal eigenform

Let  $p$  be an odd prime, and let  $\omega : \mathbf{F}_p^\times \rightarrow \mathbf{Z}_p^\times$  be the unique character (the Teichmüller lift) such that  $\omega(d) \equiv d \pmod{p}$  for every  $d \in \mathbf{F}_p^\times$ . The image of  $\omega$  is the subgroup  $_{p-1}\mu$  or roots of 1 of order dividing  $p-1$ . If a number field  $K$  contains these roots of 1, and if we fix a prime  $\mathfrak{p}|p$  of  $K$ , then we may think of  $\omega$  as taking values in  $K^\times$ .

Let  $k \in [2, p-3]$  be an even integer such that  $p|B_k$ , and put  $\varepsilon = \omega^{k-2}$ . See Anupam Saikia's contribution [12] to this volume for the proof of the following crucial result.

**THEOREM 3.7.** — *Suppose that  $p|B_k$ . There exist a normalised cuspidal eigenform  $f \in \mathfrak{S}_2(p, \varepsilon)$ ,  $f = \sum_{n \geq 0} a_n q^n$ , of weight 2, level  $p$ , and character  $\varepsilon$ , and a prime  $\mathfrak{p}|p$  of the number field  $K_f$  (0.4) generated by the  $q$ -coefficients of  $f$  such that for every prime  $l \neq p$ , the number  $a_l$  is  $\mathfrak{p}$ -integral and*

$$a_l \equiv 1 + l^{k-1} \equiv 1 + \varepsilon(l)l \pmod{\mathfrak{p}}.$$

Note that  $K_f$  is a number field (0.4) and that  $\varepsilon$  may be thought of as taking values in  $K_f^\times$ , for  $K_f$  contains  $\mathbf{Q}(\varepsilon)$ , the number field generated by the values of  $\varepsilon$  (0.4).

Now let  $\rho_{f,\mathfrak{p}}$  be the representation attached to  $f$  at  $\mathfrak{p}$ ; we show that it has a reduction  $r$  having the properties (i)–(iv) of (1.3).

#### 4. The Galois representation

Denoting the  $p$ -adic cyclotomic character by  $\tilde{\chi} : G_{\bar{\mathbf{Q}}} \rightarrow \mathbf{Z}_p^\times$  (so that  $\chi : \Delta \rightarrow \mathbf{F}_p^\times$  is the reduction of  $\tilde{\chi}$ ), and thinking of it as taking values in  $K_{f,\mathfrak{p}}^\times$ , the equality  $\det(\rho_{f,\mathfrak{p}}(\text{Frob}_l)) = l\varepsilon(l)$  (0.6) can be rewritten as  $\det \rho_{f,\mathfrak{p}} = \tilde{\chi} \cdot \varepsilon$ .

PROPOSITION 4.1. — *The representation  $\rho_{f,\mathfrak{p}}$  is simple.*

Suppose that it is not; its semisimplification is then  $\rho_1 \oplus \rho_2$ , for some characters  $\rho_i : G_{\bar{\mathbf{Q}}} \rightarrow K_{f,\mathfrak{p}}^\times$ . It follows (0.2) that each  $\rho_i$  can be written as  $\chi^{n_i}$  (for some  $n_i \in \mathbf{Z}$ ) on an open subgroup of the inertia subgroup at  $p$ , and therefore  $\rho_i = \varepsilon_i \chi^{n_i}$  for some characters of finite order  $\varepsilon_i : G_{\bar{\mathbf{Q}}} \rightarrow K_{f,\mathfrak{p}}^\times$  unramified away from  $p$ .

Thinking of the  $\varepsilon_i$  as characters of  $(\mathbf{Z}/m_i\mathbf{Z})^\times$  for some  $m_i$ , we have, for every prime  $l \neq p$ , the relations

$$\varepsilon_1(l)l^{n_1} + \varepsilon_2(l)l^{n_2} = a_l, \quad \varepsilon_1(l)\varepsilon_2(l)l^{n_1+n_2} = l\varepsilon(l).$$

From the second equation we get  $n_1 + n_2 = 1$ , and hence one of the  $n_i$  is  $\geq 1$  and the other  $\leq 0$ . The first equation then gives  $|a_l| \geq l - 1$  which, for  $l \geq 7$ , contradicts the ‘‘Riemann hypothesis’’  $|a_l| \leq 2\sqrt{l}$ , as proved by Eichler-Shimura.

Denote the ring of integer of  $K_{f,\mathfrak{p}}$  by  $\mathfrak{o}_{f,\mathfrak{p}}$ .

PROPOSITION 4.2. — *There exists a  $G_{\bar{\mathbf{Q}}}$ -stable  $\mathfrak{o}_{f,\mathfrak{p}}$ -lattice  $\Lambda \subset V_{f,\mathfrak{p}}$  such that*

$$\rho_{f,\mathfrak{p},\Lambda} \sim \begin{pmatrix} 1 & * \\ 0 & \chi^{k-1} \end{pmatrix}, \quad \rho_{f,\mathfrak{p},\Lambda} \not\sim \begin{pmatrix} 1 & 0 \\ 0 & \chi^{k-1} \end{pmatrix}.$$

We know that  $\rho_{f,\mathfrak{p}}$  is irreducible (4.1). The proposition would follow from (2.1) once we show that  $\bar{\rho}_{f,\mathfrak{p}} \sim 1 \oplus \chi^{k-1}$ , where  $\bar{\rho}_{f,\mathfrak{p}}$  is the semisimplification of  $\rho_{f,\mathfrak{p},\Lambda}$  (0.3).

Indeed, for every prime  $l \neq p$ , we know (0.6) that  $\rho_{f,\mathfrak{p}}$ , and hence the reduction  $\bar{\rho}_{f,\mathfrak{p}}$ , is unramified at  $l$ . Moreover (0.6), the trace and the determinant of  $\text{Frob}_l \in \text{Im } \rho_{f,\mathfrak{p}}$  are  $a_l$  and  $l\varepsilon(l)$  respectively. By (3.7) and

the definition of  $\varepsilon$ , these numbers are congruent respectively to  $1 + l^{k-1}$  and  $l^{k-1}$  modulo  $\mathfrak{p}$ . By the Chebotarev density theorem, the trace and the determinant of  $\bar{\rho}_{f,\mathfrak{p}}$  are  $1 + \chi^{k-1}$  and  $\chi^{k-1}$  respectively, which are the same as the trace and the determinant of the (semisimple) representation  $1 \oplus \chi^{k-1}$ . But two degree-2 semisimple representations having the same trace and the same determinant are the same (0.3), hence  $\bar{\rho}_{f,\mathfrak{p}} \sim 1 \oplus \chi^{k-1}$ , which was to be proved.

Now fix such a lattice  $\Lambda$  and let  $r = \rho_{f,\Lambda}$  be the reduction of  $\rho_{f,\mathfrak{p}}$  along  $\Lambda$ . We have seen that the representation  $r$  satisfies the conditions (i)–(iii) of (1.3); it remains to check that it satisfies (iv) — the restriction  $r|_{D_p}$  of  $r$  to a decomposition group  $D_p$  at  $p$  is semisimple.

Denote by  $F$  the residue field of  $K_{f,\mathfrak{p}}$  and by  $M$  the (2-dimensional)  $F$ -space which carries the representation  $r$ . Let  $L$  be the completion of  $\mathbf{Q}(\zeta)^+$  at the unique place above  $p$ , and  $D = \text{Gal}(\bar{L}|L)$  the decomposition group at the said place. It is sufficient to show that  $r|_D$  (or the  $D$ -module  $M$ ) is semisimple, for the index  $(D_p : D)$  is prime to  $p$ , where we have identified  $D_p$  with  $\text{Gal}(\bar{L}|\mathbf{Q}_p)$ .

**PROPOSITION 4.3.** — *The  $D$ -module  $M$  comes from a finite flat  $p$ -elementary commutative group scheme over the ring of integers  $\mathfrak{o}_L$  of  $L$ .*

There is an abelian  $\mathbf{Q}$ -variety  $A$  which is  $\mathbf{Q}$ -isogenous to  $A_f$  and such that  $\mathfrak{o}_f \subset \text{End}(A)$ , because  $K_f \subset \text{End}_{\mathbf{Q}}(A_f) \otimes \mathbf{Q}$ . The  $D$ -module  $M$  is then the “kernel”  ${}_p A$  of  $\mathfrak{p}$ , and hence contained in the  $D$ -module  ${}_p A$ , the kernel of  $p$ . As  $A$  has good reduction over  $L$  (0.7), the  $D$ -module  ${}_p A$  comes from a finite flat  $p$ -elementary commutative group scheme over  $\mathfrak{o}_L$ , namely the kernel  ${}_p \mathcal{A}$  of  $p$ , where  $\mathcal{A}$  — the néronian model of  $A$  — is the abelian  $\mathfrak{o}_L$ -scheme whose generic fibre is  $A$ . It follows that the  $D$ -module  $M = {}_p A$  comes from the schematic closure  $\mathcal{M}$  of  $M$  in  ${}_p \mathcal{A}$  (0.8).

**THEOREM 4.4.** — *The  $D$ -module  $M$  is semisimple.*

This is the same as saying that the order of  $r(D)$  is prime to  $p$ , and will follow from the existence of two distinct  $F$ -lines ( $X$  and  $M^0$ ) in  $M$  which are  $D$ -stable.

We know that  $r$ , and hence  $r|_D$ , is isomorphic to  $\begin{pmatrix} 1 & * \\ 0 & \chi^{k-1} \end{pmatrix}$ . Let  $X \subset M$  be an  $F$ -line on which  $D$  acts trivially, so that  $D$  acts via  $\chi^{k-1}$  on the  $F$ -line  $Y = M/X$ ; in particular,  $Y$  is ramified.

Let  $\mathcal{X}$  be the schematic closure of  $X$  in  $\mathcal{M}$ , so that the  $D$ -module coming from  $\mathcal{X}$  is  $X$ . As the absolute ramification index  $(p-1)/2$  of  $L$  is  $< p-1$ , the group scheme  $\mathcal{X}$  is constant (0.8). It follows that  $\mathcal{M}$  cannot be connected,

for it has the étale subgroup scheme  $\mathcal{X}$  (of order  $> 1$ ).

Now, the group scheme  $\mathcal{M}$  is an F-space scheme. Let  $M^0 \subset M$  be the sub-F[D]-module coming from the largest connected subgroup scheme of  $\mathcal{M}$ , so that the F[D]-module  $M/M^0$  comes from the largest étale quotient of  $\mathcal{M}$  and hence  $M/M^0$  is unramified.

We have  $M^0 \neq M$ , because  $\mathcal{M}$  is not connected, as we have seen. We have  $M^0 \neq 0$  because  $M/M^0$  is unramified whereas  $M$  is not (for it has the quotient  $Y$  which is ramified). For the same reason,  $M^0 \neq X$ , because  $M/M^0$  is unramified whereas  $Y = M/X$  is ramified. Thus  $X$  and  $M^0$  are two distinct D-stable F-lines in  $M$ , and hence the D-module  $M$  is semisimple, which was to be shown.

*Summary of the proof of the main theorem (1.1).* — The first step (3.7) is to construct, if  $p|B_k$ , a normalised cuspidal eigenform  $f \in \mathfrak{S}_2(\Gamma_1(p))$  such that for every prime  $l \neq p$ , the  $q$ -coefficient  $a_l(f)$  is  $\equiv 1 + l^{k-1}$  modulo a certain prime  $\mathfrak{p}|p$  of  $K_f$ . A certain reduction  $r$  (2.1) of the associated  $\mathfrak{p}$ -adic representation  $\rho_{f,\mathfrak{p}}$  gives an everywhere-unramified  $p$ -elementary abelian extension  $E|\mathbf{Q}(\zeta)$  which is Galois over  $\mathbf{Q}$  and such that  $\Delta = \text{Gal}(\mathbf{Q}(\zeta)|\mathbf{Q})$  operates on  $H = \text{Gal}(E|\mathbf{Q}(\zeta))$  via the character  $\chi^{1-k} : \Delta \rightarrow \mathbf{F}_p^\times$ . Hence  $C(\chi^{1-k}) \neq 0$ .

## 5. A weight- $k$ level-1 cuspidal eigenform

At the time of writing [16], Ribet was aware that instead of the form  $f \in \mathfrak{S}_2(p, \varepsilon)$  of (3.7), one can try using a suitable form  $g \in \mathfrak{S}_k(1)$ , as suggested by [13]. The construction of  $g$  is more conceptual and less computational than that of  $f$ . He could show that for some prime  $\mathfrak{q}|p$  of the number field  $K_g$ , the associated representation  $\rho_{g,\mathfrak{q}}$  has a reduction  $r$  satisfying properties (i)–(iii) of (1.3). However, the techniques of the day were insufficient to prove that the said reduction had the property (iv).

As pointed out by C. Khare [11], it has now become possible to show that  $r$  satisfies (1.3)(iv), thanks to the work of Faltings-Jordan [9] and Fontaine-Laffaille [10]. We show the existence of  $g$  and prove that there is indeed an  $r$  satisfying (1.3)(iv) as well.

**PROPOSITION 5.1.** — *Suppose that  $p|B_k$  for some even  $k \in [2, p-3]$ . There exist a normalised weight- $k$  level-1 cuspidal eigenform  $g = \sum_{n>0} a_n(g)q^n$  in  $\mathfrak{S}_k(1)$  and a prime  $\mathfrak{q}|p$  of the number field  $K_g$  generated by the  $q$ -coefficients of  $g$  such that for every prime  $l \neq p$ , the number  $a_l(g)$  is  $\mathfrak{q}$ -integral and*

$$a_l(g) \equiv 1 + l^{k-1} \pmod{\mathfrak{q}}.$$

Recall that the Eisenstein series  $G_j = (-B_j/2j) + \sum_{n>0} \sigma_{j-1}(n)q^n$ ,

where  $\sigma_i(n) = \sum_{d|n} d^i$ , is in  $\mathfrak{M}_j(1)$  for every even integer  $j > 2$ ; our  $k$  is  $> 2$ , for the denominator of  $B_2$  is 1. Notice that  $a_l(G_k) = 1 + l^{k-1}$  for every prime  $l$ . Also recall that there is a unique normalised cuspidal eigenform  $\Delta \in \mathfrak{S}_{12}(1)$ , with  $q$ -expansion

$$q \prod_{n>0} (1 - q^n)^{24} = \sum_{n>0} \tau(n) q^n.$$

The graded  $\mathbf{C}$ -algebra of level-1 modular forms  $\mathfrak{M}(1)$  is the polynomial ring  $\mathbf{C}[G_4, G_6]$ , and the graded ideal  $\mathfrak{S}(1) \subset \mathfrak{M}(1)$  of level-1 cuspidal forms is principal, generated by  $\Delta$  [8, p. 88].

Following a suggestion of K. Joshi, we therefore look for a  $g$  of the type  $g = g_1 \Delta + \cdots + g_n \Delta^n$  for a suitable  $n > 0$ , where  $g_i = a_i G_4^{c_i} G_6^{d_i}$  ( $a_i \in \mathbf{Z}$ ,  $c_i, d_i \in \mathbf{N}$ ,  $4c_i + 6d_i + 12i = k$ ). Notice that every even integer  $> 2$  can be written as  $4c + 6d$  for some  $c, d \in \mathbf{N}$ .

The existence of a  $g \in \mathfrak{S}_k(1)_{\mathbf{Z}}$  such that  $g \equiv G_k \pmod{p}$  now follows from the fact that the constant terms  $-B_4/8$  and  $-B_6/12$  of  $G_4, G_6$  have numerators 1,  $-1$  respectively, and that if  $f_1, f_2 \in \mathfrak{M}_k(1)$  have the same first few (roughly  $k/12$ )  $q$ -coefficients, then  $f_1 = f_2$ , because the space  $\mathfrak{M}_k(1)$  is finite-dimensional. For example,  $g = \Delta$  would do if  $p = 691$  and  $k = 12$ .

But we need a normalised *eigenform*, not just a cuspidal form. This is furnished by the following lemma of Deligne-Serre, which implies the existence of a number field  $L'$ , a prime  $\mathfrak{q}|p$  of  $L'$ , and a normalised eigenform  $g \in \mathfrak{S}_k(1)_{L'}$  with  $\mathfrak{q}$ -integral  $q$ -coefficients, such that the congruence  $a_l(g) \equiv 1 + l^{k-1} \pmod{\mathfrak{q}}$  holds for every prime  $l \neq p$ .

LEMMA 5.2 [7, p. 522]. — *Let  $A$  be a discrete valuation ring,  $\mathfrak{m}$  its maximal ideal, and  $L$  its field of fractions. Let  $M$  be a free  $A$ -module of finite rank, and  $\mathfrak{h}$  a set of commuting  $A$ -endomorphisms of  $M$ . Let  $f \in M/\mathfrak{m}M$  be an eigenvector for every  $T \in \mathfrak{h}$ , of eigenvalue  $a_T \in A/\mathfrak{m}$ . There exists a discrete valuation ring  $A'$  containing  $A$ , whose field of fractions is a finite extension of  $L$  and whose maximal ideal  $\mathfrak{m}'$  restricts to  $\mathfrak{m}$ , and a common eigenvector  $f' \in M \otimes_A A'$  for  $\mathfrak{h}$  such that the eigenvalue  $a'_T \in A'$  of  $f'$  is  $\equiv a_T \pmod{\mathfrak{m}'}$  for every  $T \in \mathfrak{h}$ .*

See Anupam Saikia's contribution to this volume [12, 4.2] for a proof.

To complete the proof of (5.1), apply this lemma with  $A = \mathbf{Z}_{(p)}$ ,  $M = \mathfrak{S}_k(1)_{\mathbf{Z}_{(p)}}$ ,  $\mathfrak{h}$  the set of Hecke operators  $T_l$  for primes  $l$ , and  $f = \bar{g}$ , the reduction of  $g$  modulo  $p$ , which is an eigenform in  $M/pM = \mathfrak{S}_k(1)_{\mathbf{F}_p}$ , the eigenvalue for  $T_l$  being  $1 + l^{k-1} \in \mathbf{F}_p$  for every  $l \neq p$ , because  $g \equiv G_k$ .

Recall that to  $g$  and  $\mathfrak{q}$  as in (5.1) Deligne [5] attaches a degree-2 representation  $\rho_{g,\mathfrak{q}}$  of  $G_{\bar{\mathbf{Q}}}$  over the completion  $K_{g,\mathfrak{q}}$  of  $K_g$  at  $\mathfrak{q}$ ; it is unramified at every prime  $l \neq p$ ; the trace of  $\rho_{g,\mathfrak{q}}(\text{Frob}_l)$  is  $a_l(g)$ , the determinant  $l^{k-1}$ .

PROPOSITION 5.3. — *The representation  $\rho_{g,\mathfrak{q}}$  is simple.*

The proof is the same as for the representation  $\rho_{f,\mathfrak{p}}$  in (4.1), using the Ramanujan-Petersen estimate “ $|a_l(g)| \leq 2\sqrt{l^{k-1}}$  for every prime  $l \neq p$ ”, as proved by Deligne [5], instead of the earlier “ $|a_l(f)| \leq 2\sqrt{l}$ ”.

PROPOSITION 5.4. — *The representation  $\rho_{g,\mathfrak{q}}$  has a reduction  $r$  such that*

$$r \sim \begin{pmatrix} 1 & * \\ 0 & \chi^{k-1} \end{pmatrix}, \quad r \not\sim \begin{pmatrix} 1 & 0 \\ 0 & \chi^{k-1} \end{pmatrix}.$$

This follows from (2.1) much in the same way as (4.2) does, because  $\bar{\rho}_{g,\mathfrak{q}} \sim 1 \oplus \chi^{k-1}$ , since the trace of every reduction is  $\equiv 1 + \chi^{k-1} \pmod{\mathfrak{q}}$  and the determinant is  $\equiv \chi^{k-1}$ .

PROPOSITION 5.5. — *The restriction  $r|_{D_p}$  of  $r$  to the decomposition group  $D_p$  at  $p$  is semisimple; in fact, the order of  $r(D_p)$  is prime to  $p$ .*

## 6. The denominators of Bernoulli numbers

Let's conclude with Witt's unpublished proof, as presented in [2], about the prime divisors of the denominators of  $B_k$ . It implies in particular that the numerator of  $B_{p-1}$  is never divisible by  $p$ .

THEOREM 6.1 (von Staudt–Clausen, 1840). — *Let  $k > 0$  be an even integer, and let  $l$  run through the primes. Then the number*

$$(1) \quad W_k = B_k + \sum_{l-1 \mid k} \frac{1}{l}$$

*is always an integer. For example,  $B_{12} + \frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{13} = 1$ .*

The British analyst Hardy says in his *Twelve lectures* (p. 11) that this theorem was rediscovered by Ramanujan “at a time of his life when he had hardly formed any definite concept of proof”.

*Proof* (Witt) : The idea is to show that  $W_k$  is a  $p$ -adic integer for every prime  $p$ . More precisely, we show that  $B_k + p^{-1}$  (resp.  $B_k$ ) is a  $p$ -adic integer if  $p-1 \mid k$  (resp. if not).

For an integer  $n > 0$ , let  $S_k(n) = 0^k + 1^k + 2^k + \dots + (n-1)^k$ . Comparing the coefficients on the two sides of

$$1 + e^T + e^{2T} + \dots + e^{(n-1)T} = \frac{e^{nT} - 1}{T} \frac{T}{e^T - 1},$$

we get  $S_k(n) = \sum_{m \in [0, k]} \binom{k}{m} \frac{B_m}{k+1-m} n^{k+1-m}$ . To recover  $B_k$  from  $S_k(n)$ ,

it is tempting to take the limit  $\lim_{n \rightarrow 0} S_k(n)/n$ , which doesn't make sense in the archimedean world. If, however, we make  $n$  run through the powers  $p^s$  of a fixed prime  $p$ , then,  $p$ -adically,  $p^s \rightarrow 0$  as  $s \rightarrow +\infty$ , and

$$(2) \quad \lim_{s \rightarrow +\infty} S_k(p^s)/p^s = B_k.$$

Let us compare  $S_k(p^{s+1})/p^{s+1}$  with  $S_k(p^s)/p^s$ . Every  $j \in [0, p^{s+1}[$  can be uniquely written as  $j = up^s + v$ , where  $u \in [0, p[$  and  $v \in [0, p^s[$ . Now,

$$\begin{aligned} S_k(p^{s+1}) &= \sum_{j \in [0, p^{s+1}[} j^k = \sum_{u \in [0, p[} \sum_{v \in [0, p^s[} (up^s + v)^k \\ &\equiv p \left( \sum_v v^k \right) + kp^s \left( \sum_u u \sum_v v^{k-1} \right) \pmod{p^{2s}} \end{aligned}$$

by the binomial theorem. As  $\sum_v v^k = S_k(p^s)$  and  $2 \sum_u u = p(p-1) \equiv 0 \pmod{p}$ , we get

$$S_k(p^{s+1}) \equiv p S_k(p^s) \pmod{p^{s+1}},$$

where, for  $p = 2$ , the fact that  $k$  is even has been used. Dividing throughout by  $p^{s+1}$ , this can be expressed by saying that

$$\frac{S_k(p^{s+1})}{p^{s+1}} - \frac{S_k(p^s)}{p^s} \in \mathbf{Z}_{(p)}$$

is a  $p$ -adic integer, and therefore

$$\frac{S_k(p^r)}{p^r} - \frac{S_k(p^s)}{p^s} \in \mathbf{Z}_{(p)}$$

for any two integers  $r > 0$ ,  $s > 0$ , since  $\mathbf{Z}_{(p)}$  is a subring of  $\mathbf{Q}$ . Fixing  $s = 1$  and letting  $r \rightarrow +\infty$ , we see that  $B_k - S_k(p)/p \in \mathbf{Z}_{(p)}$ , in view of (2). We need

LEMMA 6.2. —  $S_k(p) = \sum_{j \in [1, p[} j^k \equiv \begin{cases} -1 \pmod{p} & \text{if } p-1 \mid k \\ 0 \pmod{p} & \text{otherwise.} \end{cases}$

This is clear if  $p-1 \mid k$ . Suppose not, and let  $g$  be a generator of  $(\mathbf{Z}/p\mathbf{Z})^\times$ . We have  $g^k - 1 \not\equiv 0$ , whereas

$$(g^k - 1) \left( \sum_{j \in [1, p[} j^k \right) \equiv (g^k - 1) \left( \sum_{t \in [0, p-1[} g^{tk} \right) \equiv g^{(p-1)k} - 1 \equiv 0.$$

It follows that  $B_k + p^{-1} \in \mathbf{Z}_{(p)}$  if  $p-1 \mid k$  and  $B_k \in \mathbf{Z}_{(p)}$  otherwise. In either case, the number  $W_k(1)$ , which can be written as

$$W_k = \begin{cases} (B_k + p^{-1}) + \sum_{l \neq p} l^{-1} & \text{if } p-1 \mid k \\ (B_k) + \sum_l l^{-1} & \text{otherwise,} \end{cases}$$

(where  $l$  runs through the primes for which  $l-1 \mid k$ ) turns out to be a  $p$ -adic integer for every prime  $p$ . Hence  $W_k \in \mathbf{Z}$ , as claimed.

John Coates remarked at the workshop that the analogue of (6.1) for a totally real number field  $F$  (other than  $\mathbf{Q}$ ) is an open problem; even a weak analogue would imply Leopoldt's conjecture for  $F$ .

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